Question 53 (21 July 2009, Q4).
Suppose you sign a *forward* contract with maturity of one year on an asset which does not pay any income until maturity. Suppose the current price of the underlying asset is €40, while the continuously compounding interest rate is 10 percent.

1. What is the *forward* price and the initial value of the *forward* contract?
2. Six months later the price of the underlying asset has reached €45, while the interest rate is still 10 percent. What is now the *forward* price for the same delivery date? What is the value of the *forward* contract you signed six months before?

Solution.
The price of a *forward* contract when it is signed, F_0, is equal to its delivery price, K_0. Therefore, it is given by the following formula,

$$F_0 \;=\; K_0 \;=\; S_0\, e^{rT},$$

where r is the continuously compounding interest rate (on a per year basis), T is the time-to-maturity (in years) for the contract, and S_0 is the initial price of the underlying asset.

In the specific case we have that,

$$F_0 \;=\; 40\, e^{0.1\times 1} \;=\; 44.21.$$

Since the initial *forward* price and delivery price are equal, the initial economic value of the contract is zero for both counter-parties so that no money is exchanged at the time the contract is signed. However, the economic value of a *forward* contract changes overtime, following the changes in the price of the underlying asset. In particular, at time t the *forward* price can be different from the delivery price. The *forward* price is now equal to,

$$F_t \;=\; S_t\, e^{r(T-t)}.$$

Hence, after six months, the *forward* price is

$$F_t \;=\; 45\, e^{0.1\times 0.5} \;=\; 47.31,$$

where $S_t = 45$ is the new price for the underlying asset. The economic value of the existing *forward* contract is equal to the present value of the difference between the *forward* price and the delivery price,

$$f_t \;=\; (F_t \;-\; K_0)\, e^{-r(T-t)},$$

that is,

$$f_t \;=\; (47.31 \;-\; 44.21)\, e^{-0.1\times 0.5} \;=\; 3.10 \times 0.95 \;=\; 2.945.$$

Question 55 (13 October 2010, Q3).
Consider a *futures* contract on gold which matures in 6 months. Assume the current spot price of gold is \$600 per once. Let the interest rate, with a monthly compounding frequency, be equal to 6% (on a per annum basis). Assume the current futures price is \$617 per once.

1. Show there exists an arbitrage opportunity.
2. Explain how you can build and arbitrage portfolio.
3. Indicate what profit can be gained using such an arbitrage portfolio.

Solution.

1. To check the existence of an arbitrage opportunity there are alternative methods. The easiest consists of calculating the equilibrium futures price. In the specific case the interest rate assumes that interests are compounded with a monthly frequency. We know that if the frequency of capitalization is m times per year and the corresponding interest rate on a per annum basis is r_m, then the equilibrium futures price is the following,

$$F_t = S_t \left(1 + \frac{r_m}{m}\right)^{(T-t)\cdot m},$$

where $T - t$ is measured in years. To check such a formula, you can proceed as usual, comparing two alternative investment portfolios. In the former we purchase an once of gold in the spot market, in the latter we take a long position in the futures contract and invest the present value of the delivery price, $F_t/(1 + r_m/m)^{(T-t)\cdot m}$, at the interest rate r_m. The two portfolios yield the same payoffs at maturity (i.e. in T) and hence must cost the same in t. The non-arbitrage condition pins down the equilibrium futures price. In the specific case this is

$$F_t = \$600\,(1 + 0.06/12)^{\frac{12}{2}} = \$600\,(1.005)^6 = \$618.23$$

In brief, the futures contract is underpriced.

2. We should apply the following arbitrage strategy: purchase the futures contract and sell spot gold. More precisely, assume we take a long position on the futures, to short sell an once of gold and to invest its proceeds at the interest rate $r_m = 6\%$ for six months.

3. In the next Table we report the initial investment and final payoffs.

Position	Investment	*Pay-off*
Invest \$600	\$600	\$618.23
Short sale 1 once	-\$600	-S_T
Buy the *futures*	\$0	S_T - \$617
Total	\$0	\$1.23

Hence the certain, final gain is \$1.23, while the initial investment is null.

Question 58 (9 September 2013, Q2).
Assume the spot price for a *commodity* is \$100, while the price of a *futures* contract with maturity of 1 year is \$105. Assume the continuously compounding interest rate for the yearly maturity is 10 percent.

1. Show that there is an inconsistency in the *futures* price.

2. Describe an arbitrage strategy which allows to exploit such inconsistency and calculate the corresponding arbitrage profits.

Solution.

1. Consider that the equilibrium value for the futures price is

$$F_t = S_t e^{r(T-t)} = \$100\, e^{0.1} = \$110.52.$$

   As the futures contract is currently underpriced, one could short-sell the commodity, investing the proceeds into a risk-free asset for one year, and take a long position in the futures contract. At maturity the arbitrageur shall employ the proceeds of the risk-free investment to purchase the commodity at the delivery price.

   Indeed, in efficient markets arbitrageurs will massively purchase the futures contract. Then, the corresponding price should augment as consequence of the resulting buying pressure.

2. Consider the following Table, which describes the portfolio's *payoffs* at maturity.

Position	Investment	*Payoff*
Short *commodity*	-\$100	- S_T
Long *futures*	0	S_T - \$105
Long risk-free asset	\$100	\$110.52
Total	\$0	\$5.52

   Profits are equal to \$5.52.