Question 33 (13 September 2010, Q3).
A 5 year coupon bond pays a yearly coupon rate of 8%. Assume that the term structure is flat and the yield rate, y, is 10 percent.

1. What is the bond price, B?

2. What is its modified duration, MD?

3. Estimate the price impact ($\widehat{\Delta B}$) of an increase in the yield rate of 100 basis points, $\Delta y = 1\%$, using the modified duration.

4. Let ΔB denote the actual price impact of an increase in the yield rate of 100 basis point, $\Delta y = 1\%$. What is the approximation error you commit when you estimate the price impact using the modified duration ($\Delta B - \widehat{\Delta B}$)?

Solution.

1. The bond price is,

$$\begin{aligned} B &= \frac{8}{0.10}\left[1 - \frac{1}{(1+0.10)^5}\right] + \frac{100}{(1+0.10)^5} \\ &= 80\ [1 - 0.6209] + \frac{100}{1.61051} \\ &= 30.3263 + 62.0921 = 92.4184. \end{aligned}$$

2. The duration is

$$\frac{1}{92.4184}\left(\frac{8}{1+0.1} \times 1 + \frac{8}{(1+0.1)^2} \times 2 + \frac{8}{(1+0.1)^3} \times 3 + \frac{8}{(1+0.1)^4} \times 4 + \frac{108}{(1+0.1)^5} \times 5\right),$$

that is

$$\frac{1}{92.4184}\left(7.2727 \times 1 + 6.6116 \times 2 + 6.0105 \times 3 + 5.4641 \times 4 + 67.0595 \times 5\right) = 395.68813/92.4184 = 4.281$$

The modified duration is hence

$$MD = \frac{4.266}{(1+0.1)} = 3.8922.$$

3. The estimated price variation is

$$\widehat{\Delta B} = -B\, MD\, \Delta y.$$

Therefore,

$$\widehat{\Delta B} = -92.4184 \times 3.8922 \times 0.01 = -3.5971.$$

4. If the yield rate is $y = 11\%$, the actual price becomes

$$\begin{aligned} B &= \frac{8}{0.11}\left[1 - \frac{1}{(1+0.11)^5}\right] + \frac{100}{(1+0.11)^5} \\ &= 72.7272\ [1 - 0.5935] + \frac{100}{1.68506} \\ &= 29.5671 + 59.3451 = 88.9122. \end{aligned}$$

The actual price reduction is hence equal to $\Delta B = -3.5062$. The approximation error is hence $\Delta B - \widehat{\Delta B} = -0.909$.

Question 31 (24 June 2009, Q3).
Assume the term structure is flat. Then, indicate which and why of the following statements are true and which are false:

1. When, other things being equal, the yield rate rises, so does the duration of a bond.
2. Other things being equal, the larger the coupon rate of a bond and/or smaller its time to maturity, the larger its duration.
3. The modified duration allows to measure exactly the change in the price of a bond induced by either an increase or a reduction in the yield rate.

Solution.

1. Given the definition of duration this statement is surely wrong. In fact, a bond's duration is equal to the weighted average of the maturities of the individual *cash flows* it promises. These weights are proportional to the present value of the individual *cash flows*. The larger the yield rate, the smaller the present value of the more distant *cash flows* and hence the smaller the duration of the bond.
2. Even this statement is wrong. For a given maturity, the duration is larger the smaller the present value of the initial *cash flows*, that is the smaller the coupon rate it promises. In addition, for a given coupon rate, the larger the bond's time to maturity the larger its duration. In fact, there are more terms in the weighted average of the maturities of the bond's *cash flows*.
3. The last statement is wrong as well. In fact, the modified duration does not take into account the curvature of the function linking a bond's price to the yield rate. Taking account of such curvature requires considering a measure of convexity.

   (In fact, via Taylor's formula, we can check that

$$\Delta B \approx \frac{\partial B}{\partial y} \Delta y + \frac{1}{2} \frac{\partial B^2}{\partial y^2} \Delta y^2,$$

   where B denotes the bond's value and y the yield rate. If MD denotes the modified duration and C the convexity of the bond, we find that

$$MD = -\frac{1}{B} \frac{\partial B}{\partial y}, \qquad C = \frac{1}{B} \frac{\partial B^2}{\partial y^2}.$$

   Hence, we conclude that

$$\Delta B \approx \frac{\partial B}{\partial y} \Delta y + \frac{1}{2} \frac{\partial B^2}{\partial y^2} \Delta y^2 = -B\, MD\, \Delta y + \frac{1}{2} B\, C\, \Delta y^2.)$$

Question 3
Consider the following *zeros*, with nominal values €100 and maturity respectively equal to 1, 2, 3, 4 and 5 years. Suppose their current prices are respectively €97.087, €92.456, €86.384, €79.209 and €71.299.

(a) Calculate the current spot interest rates for the five maturities.

(b) Describe the term structure of interest rates. What is its shape?

(c) Find the one-year forward interest rate valid for next year, and the following three ones.

(d) Assuming the expectation theory holds, which conclusions can we draw on the market expectations of future spot interest rates? Motivate your answer.

Question 3

(a) Calculating the *term structure* is straightforward as we have zero-coupon bonds. Then,

$$(1 + r_{0,1}) = \frac{100}{97.087} \iff r_{0,1} = 0.03 = 3.00\%,$$

$$(1 + r_{0,2})^2 = \frac{100}{92.456} \iff r_{0,2} = 0.04 = 4.00\%,$$

$$(1 + r_{0,3})^3 = \frac{100}{86.384} \iff r_{0,3} = 0.05 = 5.00\%,$$

$$(1 + r_{0,4})^4 = \frac{100}{79.209} \iff r_{0,4} = 0.06 = 6.00\%,$$

$$(1 + r_{0,5})^5 = \frac{100}{71.299} \iff r_{0,5} = 0.07 = 7.00\%.$$

(b) The term structure is positively sloped in that for $l > j$ we see that $r_{0,l} > r_{0,j}$.

(c) We can obtain the forward rates for different maturities and years. We consider the following definition

$$(1 + r_{0,j})^j (1 + f_{j,n})^n \equiv (1 + r_{0,j+n})^{j+n},$$

where $f_{j,n}$ denotes the forward rate set today, date 0, for a contract which covers the interval of time between period j and $j + n$, whereas $r_{0,j}$ is the current spot rate for maturity j, that is relative to a contract which covers the period between period 0, today, and period j.

Since we need the one-year forward rate valid for next and the following years, setting $n = 1$ and $j = 0, 1, \ldots, 5$ we find that

$$(1 + r_{0,1})(1 + f_{1,1}) \equiv (1 + r_{0,2})^2 \iff (1 + 0.03)(1 + f_{1,1}) \equiv (1 + 0.04)^2 \iff$$
$$f_{1,1} \approx 0.05 = 5.00\%,$$

$$(1 + r_{0,2})^2 (1 + f_{2,1}) \equiv (1 + r_{0,3})^3 \iff (1 + 0.04)^2 (1 + f_{2,1}) \equiv (1 + 0.05)^3 \iff$$
$$f_{2,1} \approx 0.07 = 7.00\%,$$

$$(1 + r_{0,3})^3 (1 + f_{3,1}) \equiv (1 + r_{0,4})^4 \iff (1 + 0.05)^3 (1 + f_{3,1}) \equiv (1 + 0.06)^4 \iff$$
$$f_{3,1} \approx 0.09 = 9.00\%,$$

$$(1 + r_{0,4})^4 (1 + f_{4,1}) \equiv (1 + r_{0,5})^5 \iff (1 + 0.06)^4 (1 + f_{4,1}) \equiv (1 + 0.07)^5 \iff$$
$$f_{4,1} \approx 0.11 = 11.00\%.$$

This shows that the one-year forward rates are greater than the one-year spot rate, ie. for $j = 1, \ldots 4$ we find that

$$f_{j,1} > r_{0,1}.$$

(d) If the expectation theory holds, the forward rates are equal to the current expectations of the corresponding future spot rates. Formally,

$$f_{j,n} = E_0[\tilde{r}_{j,n}],$$

where now $\tilde{r}_{j,n}$ denotes the spot rate for maturity n prevailing in period j.

As mentioned from the term structure, $\{r_{0,1}, \ldots, r_{0,N}\}$, we can derive the forward rate using the following definition,

$$(1 + r_{0,j+n})^{j+n} \equiv (1 + r_{0,j})^j (1 + f_{j,n})^n.$$

Hence, assume that the term structure is positively sloped, as in the current example, so that the spot interest rates rise with their maturity,

$$r_{0,n} > r_{0,n'} \quad \text{for } n > n'.$$

As shown within the context of the current example, we can check that any time the term structure is upward sloping the forward rates, $f_{j,n}$, take larger values than the corresponding spot rates, $r_{0,n}$. In other words, for any maturity n, we see that

$$r_{0,n} < f_{j,n}, \quad \forall j.$$

This means that the market participants expect spot interest rates to rise. In fact, since

$$f_{j,n} = E_0[\tilde{r}_{j,n}],$$

for any maturity n we see that the current spot is smaller than the expectations of its value for any future period j, that is

$$r_{0,n} < E_0[\tilde{r}_{j,n}], \quad \forall j.$$

In particular in the numerical example, given the values for $f_{1,1}$, $f_{2,1}$, $f_{3,1}$ and $f_{4,1}$, equal respectively to 5%, 7%, 9% and 11%, it is confirmed that: i) $E_0[\tilde{r}_{1,1}] = 5\% > r_{0,1} = 3\%$; ii) $E_0[\tilde{r}_{2,1}] = 7\% > r_{0,1}$, iii) $E_0[\tilde{r}_{3,1}] = 9\% > r_{0,1}$ and iv) $E_0[\tilde{r}_{4,1}] = 11\% > r_{0,1}$.